UNIVERSAL BEHAVIOR OF THE COEFFICIENTS OF THE CONTINUOUS EQUATION IN COMPETITIVE GROWTH MODELS

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The competitive growth models involving only one kind of particles (CGM), are a mixture of two processes one with probability p and the other with probability $1-p$. The $p-$dependance produce crossovers between two different regimes. We demonstrate that the coefficients of the continuous equation, describing their universality classes, are quadratic in p (or $1-p$). We show that the origin of such dependance is the existence of two different average time rates. Thus, the quadratic $p-$dependance is a universal behavior of all the CGM. We derive analytically the continuous equations for two CGM, in $1+1$ dimensions, from the microscopic rules using a regularization procedure. We propose generalized scalings that reproduce the scaling behavior in each regime. In order to verify the analytic results and the scalings, we perform numerical integrations of the derived analytical equations. The results are in excellent agreement with those of the microscopic CGM presented here and with the proposed scalings.

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Evolving growing interfaces or surfaces can be found in many physical, chemical and biological processes. For example, in film growth either by vapour deposition or chemical deposition [1, 2], bacterial growth [3] and propagation of forest fire [4]. The resulting interface has a rough surface that is characterized through scaling of the interfacial width W defined as $W(L,t) = \left\{\langle[h_i - \langle h_i\rangle]^2\rangle^{1/2}\right\}$, where h_i is the height at the position i, $\langle h_i\rangle = \sum_{i=1}^{L^d} h_i$ is the spatial average, L is the linear size, d is the spatial dimension and $\{\}$ denote configuration averages. The general scaling relation [1] for these growing interfaces that evolves through a single model can be summarized in the form $W(L,t) \sim L^{\alpha} f(t/L^z)$, where the scaling function $f(u)$ behaves as $f(u) \sim u^{\beta}$ ($\beta = z/\alpha$), for $u \ll 1$ and $f(u) \sim$ const for $u \gg 1$. The exponent α describes the asymptotic behavior where the width saturates due to finite sizes effects, while the exponent β represent the early time regime where finite-size effects are weak. The crossover time between the two regimes is $t_s = L^z$.

The study of growth models involving one kind of particles in competitive processes (CGM) has received little attention, in spite of the fact that they are more realistic describing the growing in real materials, where usually there exist a competition between different growing processes. As an example, in a colony of bacteria growing on a substrate, a new bacteria can born near to another and stay there, move into another place looking for food or died. This "bacteria" can be thought as a particle undergoing either a deposition/evaporation process or deposition/surface relaxation.

The processes involved in the CGM could have different characteristic average time rate. Recently Shapir et al. [5] reported experimental results of surface roughening during cyclical electrodeposition dissolution of silver. Horowitz et al. [6] introduced a competitive growth model between random deposition with surface relaxation (RDSR) with probability p and random deposition (RD) with probability $1-p$, called RDSR/RD. The authors proposed that the scaling behavior is characteristic of an Edward Wilkinson (EW) equation, where the coefficient associated to the surface tension ν depends on p. The dependance of ν on p governs the transition from RDRS to RD. Using a dynamic scaling ansatz for the interface width W they found that the results are consistent provide that $\nu \propto p^2$. Also Pellegrini and Jullien [7] have introduced CGM between ballistic deposition (BD) with probability p and RDSR with probability $1-p$, called BD/RDSR. For this model Chame and Aarão Ries [8] presented a more carefully analysis in $1+1$-d and showed that there exist a slow crossover from an EW to a Kardar-Parisi-Zhang (KPZ) for any $p > 0$. They also found that the parameter p is connected to the coefficient λ of the nonlinear term of the KPZ equation by $\lambda \sim p^{\gamma}$, with $\gamma = 2.1$.

In this letter, we show that the origin of such dependance is the existence of two different average time rates. Thus, the quadratic $p-$dependance is an universal feature of all the CGM. To our knowledge this is the first time that the p-dependance on the coefficient of the continuous equations is obtained analytical from the microscopic dynamics.

In order to test our hypothesis, we derive the analytical continuous equations for the local height for the RDSR/RD and BD/RDSR models. The procedure chosen here is based on regularization and coarse-graining of the discrete Langevin equations obtained from a Kramers-Moyal expansion of the master equation

[9, 10, 11].

Lets introduce first the general treatment of this problem. Let us denote by $h_i(t)$ the height of the i-th generic site at time t. The set $\{h_i,\, i = 1, \ldots, L\}$ defines the interface. Here we distinguish between two competitive processes: A with probability p and average time of deposition τ_A, and B with probability 1-p and average time of deposition τ_B. In deposition processes with $p = 1$ the average rate of deposition is given by $\tau_0^{-1} = \{dh_i/dt\}_{(p=1)}$. If the process is made with probability p the average rate of deposition is given by $\tau_A^{-1} = \{dh_i/dt\}_{(p)} = p\,\{dh_i/dt\}_{(p=1)}$. The same hold for a process with probability $(1-p)$. Thus, the particles are deposited at an average rate

$$\tau_A = \frac{\tau_0}{p}\,, \qquad \tau_B = \frac{\tau_0}{1-p}\,. \tag{1}$$

In the average time of each process, the height in the site i increases by

$$\begin{aligned} h_i(t+\tau_A) &= h_i(t) + a_\perp\, p\, R_i^A\,, \\ h_i(t+\tau_B) &= h_i(t) + a_\perp\,(1-p)\, R_i^B\,, \end{aligned} \tag{2}$$

where R_i^A and R_i^B are the growing rules for processes A and B respectively and $a_\perp$ is the vertical lattice spacing. Expanding $h_i(t+\tau_A)$ and $h_i(t+\tau_B)$ to second order in Taylor series around τ_A and τ_B, we obtain

$$h_i(t+\tau_J) - h_i(t) \approx \frac{dh_i}{dt}\,\tau_J\,, \tag{3}$$

for the process $J = A, B$. Thus, the evolution equation for the height (in the site i) for this CGM is given by

$$\frac{dh_i}{dt} = K_i^{(1,A)} + K_i^{(1,B)} + \eta_i(t)\,, \tag{4}$$

where the first moments of the transition rate for each process [12] are

$$\begin{aligned} K_i^{(1,A)} &= \frac{a_\perp}{\tau_A}\, p\, R_i^A\,, \\ K_i^{(1,B)} &= \frac{a_\perp}{\tau_B}\,(1-p)\, R_i^B\,, \end{aligned} \tag{5}$$

and the Gaussian thermal noise $\eta_i(t)$ has zero mean and covariance

$$\{\eta_i(t)\eta_j(t')\} = a_\perp \left(K_i^{(1,A)} + K_i^{(1,B)}\right)\delta_{ij}\,\delta(t-t')\,. \tag{6}$$

In order to test our analytical result, we use two models. The first model RDSR/RD considers a mixture [6] of RDSR (process A) with probability p and RD (process B) with probability $1-p$. Lets introduce the growth rule for each process for the first model. In the RD growth model one chose a column of a lattice, at random, among L and a particle is launched until it reaches the top of the selected column. The RDSR is a variant of the RD: a particle is released from a random position but when it reaches the top of the selected column is allowed to relax to the nearest neighbor (nn) column if their height are lower that the selected one. If the height of both of the nn are lower than the selected one the relaxation takes place with equal probability to one of them. For RD, $W(L,t)$ does not depend on L, this means that the width W does not saturate due to the lack of lateral correlations. Thus, in this model: $W(t) \sim t^{\beta_{RD}}$. Moreover, the RDSR model generates lateral correlations, therefore one has $\beta_{RDSR} = 1/4$ and $\alpha_{RDSR} = 1/2$. The first moment of the transition rate for these processes are

$$\begin{aligned} K_i^{(1,A)} &= \frac{a_\perp}{\tau_A}\, p\left(\omega_i^{(2)} + \omega_{i+1}^{(3)} + \omega_{i-1}^{(4)}\right)\,, \\ K_i^{(1,B)} &= \frac{a_\perp}{\tau_B}\,(1-p)\,\omega_i^{(1)}\,, \end{aligned} \tag{7}$$

where the rules for both processes can we written as

$$\begin{aligned} \omega_i^{(1)} &= 1\,, \\ \omega_i^{(2)} &= \Theta(H_i^{i+1})\,\Theta(H_i^{i-1})\,, \\ \omega_i^{(3)} &= \left\{\tfrac{1}{2}\left[1-\Theta(H_i^{i+1})\right] + \Theta(H_i^{i+1})\right\}\left[1-\Theta(H_i^{i-1})\right]\,, \\ \omega_i^{(4)} &= \left\{\tfrac{1}{2}\left[1-\Theta(H_i^{i-1})\right] + \Theta(H_i^{i-1})\right\}\left[1-\Theta(H_i^{i+1})\right]\,. \end{aligned} \tag{8}$$

where $H_{i\pm k}^{i\pm s} = (h_{i\pm s} - h_{i\pm k})/a_\perp$, and $\Theta(z)$ is the unit step function defined as $\Theta(z) = 1$ for $z \geq 0$ and $\Theta(z) = 0$ for $z < 0$. The representation of the step function can be expanded as $\Theta(z) = \sum_{k=0}^{\infty} c_k z^k$ providing that z is smooth. In any discrete model there is in principle an infinite number of nonlinearities, but at long wavelengths the higher order derivatives can be neglected using scaling arguments, since one expect affine interfaces over a long range of scales, and then one is usually concerned with the form of the relevant terms. Thus, keeping the expansion of the step function to first order in his argument and replacing the expansion Eq. (8), Eq. (4) can be written as

$$\frac{dh_i}{dt} = \frac{a_\perp(1-p)}{\tau_B} + \frac{a_\perp p}{\tau_A}\left(1 + c_1\,\frac{\Delta^2 h_i}{a_\perp}\right) + \eta_i(t)\,, \tag{9}$$

where $\Delta^2 h_i = h_{i+1} - 2h_i + h_{i+1} \simeq a_\parallel^2\;\partial^2 h/\partial x^2\rfloor_{h_i}$, and $a_\parallel$ is the horizontal lattice spacing. Replacing the rates given by Eq. (1) in Eq. (9) and using a standard coarse-grain approach [10, 11] the continuous equation for this CGM is

$$\frac{dh}{dt} = F(p) + \nu(p)\,\frac{\partial^2 h}{\partial\, x^2} + \eta(x,t)\,, \tag{10}$$

where $h = h(x,t)$ and

$$\begin{aligned} F(p) &= \frac{a_\perp}{\tau_0}\left[(1-p)^2 + p^2\right]\,, \\ \nu(p) &= 2\,c_1\,\frac{a_\parallel^2}{\tau_0}\,p^2\,. \end{aligned} \tag{11}$$

The noise covariance is given by

$$\{\eta(x,t)\eta(x',t')\} = D(p)\;\delta(x-x')\delta(t-t')\,, \tag{12}$$

where

$$D(p) = a_{\parallel}\, a_{\perp}\, F(p)\ . \tag{13}$$

Equations (11) and Eq. (13) shows that the quadratic dependance on the coefficients of the continuous equation, arises naturally as a feature of the CGM and is due to the existence of different average time rates.

The second model is a mixture of RDSR with probability $1-p$ (B process) and ballistic deposition (BD) with probability p (A process) [8]. The evolution rules for RDSR are ω_i^j, with $j = 2, 3, 4$ [see Eq. (8)]. In the BD model, the incident particle follows a straight trajectory and sticks to the surface at the column i. The height in the column i is increased in $\max[h_i + 1, h_{i+1}, h_{i-1}]$. If this process is done with probability p (A process), the rules can be summarized as:

$$\begin{aligned}
\omega_i^{(5)} &= \Theta(H_{i+1}^i)\,\Theta(H_{i-1}^i)\ , \\
\omega_i^{(6)} &= H_i^{i+1}\left[1-\Theta(H_{i+1}^i)\right]\left[1-\Theta(H_{i+1}^{i-1})\right]\ , \\
\omega_i^{(7)} &= H_i^{i-1}\left[1-\Theta(H_{i-1}^i)\right]\left[1-\Theta(H_{i-1}^{i+1})\right]\ , \\
\omega_i^{(8)} &= \tfrac{1}{2}\,\delta(H_{i-1}^{i+1},0)\,\left\{H_i^{i+1}\left[1-\Theta(H_{i+1}^i)\right]\right. \\
&\qquad \left. + H_i^{i-1}\left[1-\Theta(H_{i-1}^i)\right]\right\}\ ,
\end{aligned} \tag{14}$$

where $\delta(z,0) = \Theta(z) + \Theta(-z) - 1$ is the Kronecker delta. Following the steps leading to Eq. (10) the evolution equation for this process can be written as:

$$\frac{dh}{dt} = F(p) + \nu(p)\,\frac{\partial^2 h}{\partial x^2} + \lambda(p)\left(\frac{\partial h}{\partial x}\right)^2 + \eta(x,t) \tag{15}$$

where

$$\begin{aligned}
F(p) &= \frac{a_\perp}{\tau_0}\left[(1-p)^2 + c_0^2\,p^2\right]\ , \\
\nu(p) &= \frac{a_\parallel^2}{\tau_0}\left[\tfrac{1}{2}\,p^2(1-c_0-2c_0c_1) + 2c_1(1-p)^2\right], \\
\lambda(p) &= \frac{a_\parallel^2}{\tau_0\,a_\perp} p^2\,c_1\,(5-4c_0-c_1)\ .
\end{aligned} \tag{16}$$

The covariance of noise and $D(p)$ is given by Eq. (12) and Eq. (13), respectively. Notice that we have to change p by $1-p$ in all the above equations for RDSR, because in the first model RDSR is a kind A process and now is a kind B process. Equation (16) shows again that quadratic dependance on the coefficients of the continuous equation. The quadratic dependance of λ on p, found by Chame and Aarão Reis [8], is a general feature of the CGM.

As both models have an EW behavior, it is expected that in that regime the following generalized scaling ansatz [6, 14],

$$W^2(p,L,t) \sim L^{2\alpha}\,[D(p)/\nu(p)]\,f\left(\nu(p)\,t/L^z\right)\ , \tag{17}$$

where $f(u) \sim u^{2\beta}$ for $u \ll 1$ and $f(u) \sim$ const for $u \gg 1$. Moreover, the second model is represented by a mixture of EW and KPZ universality classes. In the early time regime $W(t) \sim t^{\beta_{RDSR}}$, while a crossover to a KPZ, with $\beta_{KPZ} = 1/3$ and $\alpha \equiv \alpha_{KPZ} = 1/2$, is expected in the intermediate regime before the saturation. Thus, for the KPZ regime we propose the following generalization [15] of the scaling behavior of the width

$$W^2(p,L,t) \sim L^{2\alpha}[D(p)/\nu(p)]\,f\left(\lambda(p)\sqrt{D(p)/\nu(p)}\,t/L^z\right)\ , \tag{18}$$

where $z = 3/2$, and $f(u) \sim u^{2\beta_{KPZ}}$ for $u \ll 1$ and $f(u) \sim$ const for $u \gg 1$.

In order to test our analytical result and the proposed scalings, we perform a numerical integration of Eq. (10) and Eq. (Notice that in order to numerically integrate the continuous equation, we do need a continuum representation of the

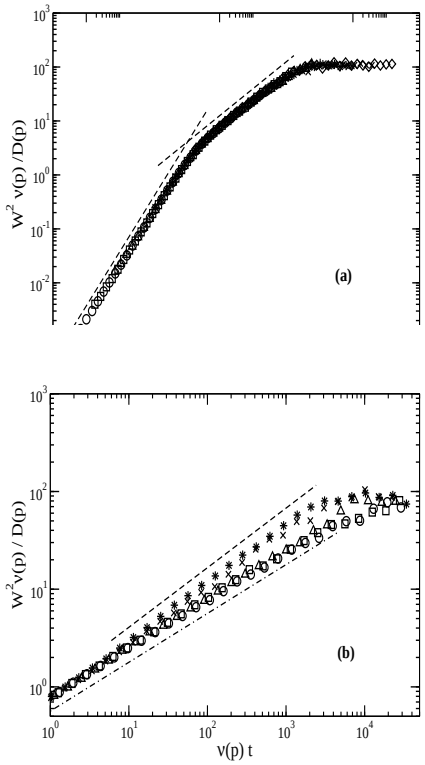

FIG. 1: (a) Log-log plot of $W^2\nu(p)/D(p)$ for the RDSR/RD model as function of $\nu(p)\,t$ for $L = 128$. The different symbols represent different values of p, $p = 0.04$ ($\circ$), $p = 0.08$ ($\Box$), $p = 0.016$ ($\triangle$), $p = 0.32$ ($+$), and $p = 0.64$ ($*$). Here we used $C = 2.58$ and $b = 0.2$ as parameters of the Θ-function representation. The dashed lines are used as guides to show the RD regime with $2\beta = 1$ and the EW regime with $2\beta = 0.5$. (b) Log-log plot of $W^2\nu(p)/D(p)$ for the BD/RDSR model as function of $\nu(p)\,t$ for $L = 1024$. The symbols represent the same as in Fig.(1a). Here we used $C = 0.18$ and $b = 0.5$. The collapse of the curves at the earlier stage clearly shows the EW behavior ($2\beta = 0.5$). After this stage the curves split and undergoes a slow crossover to the KPZ behavior ($2\beta = 0.66$). The dashed lines are used as guides to show the EW regime with $2\beta = 1/2$ and the KPZ regime. The slope showed here is $2\beta = 0.61$.

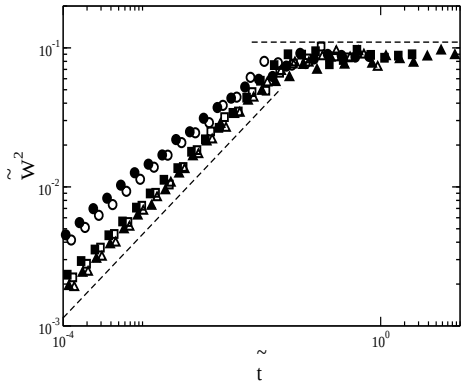

FIG. 2: Log-log plot of $\tilde{W}^2$ as function of $\tilde{t}$ as defined in the text, for $p = 0.16$ ($\circ$), $p = 0.32$ ($\Box$) and $p = 0.64$ ($\triangle$). The empty symbols correspond to $L = 512$ and the filled ones to $L = 1024$. The collapse of the curves on the saturation regime using $z = 3/2$ shows that the curves saturates with a KPZ behavior as expected.

Θ-function to numerically compute the coefficients c_0 and c_1 related to the ones of the continuous equations. To perform the numerical integration, we chose the shifted hyperbolic tangent [16] as the continuous representation of Θ-function defined as $\Theta(x) = \{1 + \tanh[C(x + b)]\}/2$, where b is the shift and C is a parameter that allows to recover the Θ in the limit $C \to \infty$. The numerical integration was made in short lattices using a discretized version of the continuous equations Eq.(10) and Eq. (15). The results in large systems and the details of the integration are beyond the scope of this letter and will be published elsewhere.

For the first model, Horowitz et al. [6] presented their data from simulations plotting the scaling relation $W/L^{\alpha}p^{-\delta}$ vs $t/L^{z}p^{-y}$. Clearly, their $\delta = 1$ and $y = 2$ is related to our $\nu(p)$ and $D(p)$ [see Eq. (11) and Eq. (13)]. In Fig. 1(a) we plot $W^2\,\nu(p)\,/D(p)$ as function of $\nu(p)\,t$ for the that model for different values of p and $L = 128$. This figure represent the same as in [6] after coarse-graining. The agreement with the results of our numerical integration, the numerical simulation [6] and the scaling presented in Eq. (17) is excellent. On the other hand, for the second model, Chame and Aarão Reis [8] did not present the result for W. They studied the crossover from EW to KPZ using an indirect method because of the slow convergence of the discrete model to KPZ behavior. The crossover is well represented in our Fig. 1(b), where we plot the same as in Fig. 1(a) but for the second model. It is clear the collapse of the curves in the EW regime. In the intermediate regime the KPZ behavior appears thus, it is expected that Eq. (18) holds in that regime. In Fig. 2 we plot $\tilde{W}^2 = W^2\,\nu(p)/[L^{2\alpha}D(p)]$ as function of $\tilde{t} = \lambda(p)\sqrt{D(p)/\nu(p)}\;t/L^{z}$ for three different values of p using $z = 3/2$. As p increases, the KPZ behavior appears earlier, but independent of p all the curves saturate as a KPZ. The agreement with Eq. (18) is excellent in the saturation regime. The departure in the intermediate regime is due to a slow crossover to the KPZ and to finite size effects.

Finally, notice that the quadratic dependence of the coefficients of the continuous equation on p is independent of the CGM considered, because it is due to two different rates of deposition given by Eq. (1). This dependence is totally generally, as shown from Eq. (1) to Eq. (7).

In summary, we demonstrate that the coefficient of the continuous equation have quadratic dependance on p. This feature is universal for all the CGM model and is due to the competition between different average time rate. We propose generalized scaling for the model that reproduce the scaling behavior in each regime. The numerical integration of the continuous equation are in excellent agreement with the propose scalings and the numerical simulation of the models.

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